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Modeling the constitutive behavior of layered composites with evolving cracks

A.A. Caiazzo [a], F. Costanzo [b,*]

[a] *Materials Sciences Corporation, Suite 250, 500 Office Center Drive, Fort Washington, PA 19034, USA*

[b] *Department of Engineering Science and Mechanics, The Pennsylvania State University, 212 Earth and Engineering Sciences Building, University Park, PA 16802, USA*

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Abstract

In this study, it is shown that a computational procedure, termed "discrete damage space homogenization method" (DDSHM), can accurately predict the constitutive response of layered composite materials containing growing cracks. The effective constitutive law for a specific layered composite architecture, as defined by the DDSHM, was integrated into the ABAQUS commercial finite element program using the user-defined material feature. Calculations were performed to show correlation with experimental data on flat laminates and curved beam elements and to illustrate the computational efficiency of the method for general analysis of composite materials with growing cracks. Results show that given the basic information about the fracture toughness of the material, the DDSHM is able to predict important material parameters, including the load at initiation of cracking, damage growth rate, and the resulting effect on the macroscopic stiffness.

Keywords: Homogenization; Micromechanics; Evolving damage; Composites

1. Introduction

Experience has shown that it can be very useful to have analysis methods to assess how details of the constituents and microstructure affect the resulting composite material behavior. The reason for this is that there are far too many material parameters from which one may choose to *engineer* a better material by experiment alone. Several of the mentioned analysis methods have been formulated, and all of them can be said to form the so-called homogenization theory. The latter provides the formal framework for the engineering analysis of effective properties of a composite material, i.e., the properties that a homogeneous material would have if it were to behave (in an average sense) like the material with discrete microstructure.

* Corresponding author. Tel.: +1-814-865-2030; fax: +1-814-863-7967.

E-mail addresses: aacmsc2@penn.com (A.A. Caiazzo), costanzo@engr.psu.edu (F. Costanzo).

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In early examples of homogenization theories, such as the so-called *classical approximation* for the analysis of an elastic composite body, the classical field equations of elasticity are assumed to be valid for a composite material body with *effective* properties replacing the *heterogeneous* properties. This concept can be extended to the study of conduction, swelling, viscoelasticity and diffusion (cf. Hashin, 1983). In principle and with some modification, these concepts can be also be extended to include composite materials with growing damage. Specific models for predicting the effective properties of composite materials with stationary or evolving damage can be characterized as (i) phenomenological or (ii) physically based.

In phenomenological modeling approaches, the effects of microdamage are generally included in the constitutive relations via a tensor valued average representation of the extent of microdamage. The overall effect of the damage is then accounted for by postulating how the thermodynamic potentials of the material depend on the chosen damage variable. These approaches have shown good correlation with experimental data (cf. Allen et al., 1994). However, since only the average response is obtained from both the model and experimental tests, and since many different microfields can yield identical averages, the correlation of results does not necessarily imply that the model is providing accurate predictions of how the microstructure (damage) has evolved. Furthermore, while it may (or may not) be possible to characterize the constitutive relation in an *average* sense using a continuum damage mechanics (CDM) model, this phenomenological approach rarely provides sufficient detail on what *caused* the damage state to evolve in any given way.

Knowledge of which microstructural detail is the "weak link" in a material can be very valuable because it may allow one to engineer a better composite. This justifies efforts trying to derive effective constitutive equations of composites alternative to phenomenological approaches. The basic idea is to be able to derive effective constitutive equations which are not based on an average representation of damage, but rather are based on fracture mechanics. In this case, for example, one would need to evaluate the Griffith type [1] fracture criterion in a physically based model for the effective response functions of elastic composite materials using techniques such as the *J*-integral (cf. Rice, 1968) or the virtual crack closure technique (VCCT) (cf. Rybicki and Kanninen, 1977). Regardless of how the local fracture criterion is evaluated, to practically apply this approach one must use a *global–local* (GL) analysis. The latter consists in conducting a structural analysis of a composite component, say using the finite element method (FEM), and *calculating* the constitutive response at every integration point by a second analysis whose purpose is to determine the evolution of the microstructure. The term GL simply reflects the nature of this approach which is essentially a two-phase process where one set of calculations are *embedded* into another set of calculations.

Previous publications by the authors (Costanzo et al., 1996; Caiazzo and Costanzo, 2000a,b) outlined and discussed the implementation of a theory for deriving the constitutive and evolution equations for composite materials with growing cracks. This physically based analysis method has been termed a discrete damage space homogenization method (DDSHM) [2] for reasons to be discussed in subsequent sections. It was offered that this procedure is a computationally efficient alternative to the GL approach for modeling the effective behavior of composite materials with growing cracks. The purpose of this paper is to present a comparison between results obtained by the DDSHM and results obtained with other methods as well as experimental studies. The paper also intends to illustrate the overall computational efficiency of the DDSHM.

[1] The well-known Griffith (1921) criterion is an energy balance which states that crack extension is possible only when the energy supplied to the system by loads is greater than the energy required to create new crack surfaces.

[2] A detailed description of the DDSHM is rather lengthy and therefore is not given in this paper. For such a description, see Caiazzo and Costanzo (2000a).

2. A review of the discrete damage space homogenization method

The DDSHM is a method for the practical implementation of the formal homogenization theory presented by Costanzo et al. (1996). This theory concerns the determination of the effective constitutive equations of composites with evolving microstructure in the forms of plasticity and microcracks. For the sake of conciseness, this section only contains a review of the DDSHM formulation and the theory behind it. Details on the formulation of the DDSHM method can be found in the work of Caiazzo and Costanzo (2000a,b). However, before proceeding any further, it is important to note that the present discussion is limited to the type of composites which will be discussed in the following examples, that is, linear elastic composites with growing microcracks.

Formally, the theory of Costanzo et al. (1996) delivers effective constitutive equations conforming to the thermodynamic theory of irreversible processes (Halphen and Nguyen, 1975; Germain et al., 1983; Maugin, 1992). In particular, the effective behavior of the composite is governed by two scalar functions: a thermodynamic and a dissipation potential. The thermodynamic potential, namely the Helmholtz free energy [3], is denoted by H and is a function of the selected *macroscopic state variables*:

$$H(t) = H\big(E_{ij}(t); \lambda_1(t), \ldots, \lambda_N(t)\big), \tag{1}$$

where $E_{ij}(t)$ is the small strain tensor and $\lambda_1(t), \ldots, \lambda_N(t)$ are N internal state variables (ISVs), and t is time. The actual constitutive equations are then derived by simply differentiating H with respect to the state variables (Germain et al., 1983), that is,

$$\Sigma_{ij} = \frac{\partial H}{\partial E_{ij}}, \qquad G_i = -\frac{\partial H}{\partial \lambda_i}, \tag{2}$$

where Σ_{ij} is the macroscopic stress tensor and G_i is the thermodynamic force promoting the growth of the ith ISV. The dissipation potential is denoted by Ω and is a function of the generalized thermodynamic forces G_i (Edelen, 1974; Halphen and Nguyen, 1975; Germain et al., 1983):

$$\Omega = \Omega(G_i), \quad i = 1, \ldots, N. \tag{3}$$

Again, the actual evolution equations of the material are obtained by differentiating Ω with respect to the generalized forces G_i, i.e.,

$$\dot{\lambda}_i = \frac{\partial \Omega}{\partial G_i}. \tag{4}$$

In a two-dimensional (2D) context, the ISVs can be chosen to represent the length of the N cracks present in the selected representative volume element (RVE). In this case, G_i represents the energy release rate for the ith crack and, using the principles of linear elastic fracture mechanics, a dissipation potential of the following form can be used (Maugin, 1992):

$$\Omega = \tfrac{1}{2}\eta_i \langle G_i - G_i^{\mathrm{cr}} \rangle^2, \quad i = 1, \ldots, N, \tag{5}$$

where η_i and G_i^{cr} measure the materials resistance to crack growth, and where $\langle \cdot \rangle$ denotes the positive part operator defined by

$$\langle \phi \rangle = \begin{cases} 0 & \text{if } \phi \leqslant 0, \\ \phi & \text{if } \phi > 0, \end{cases} \quad \phi \in \mathbb{R}. \tag{6}$$

[3] The Helmholtz (Gibbs) free energy is used in the case of a strain (stress) formulation.

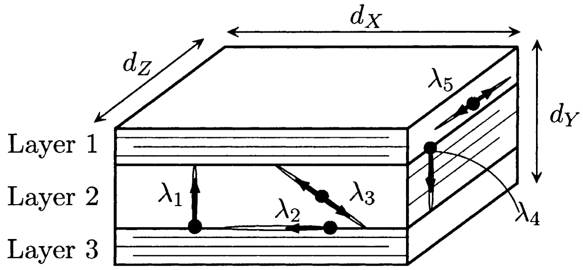

Fig. 1. A representative volume element for layered composites with growing cracks.

In order to practically use the theory outlined above, H must be evaluated for all possible values of the state variables, i.e., for an infinite number of thermodynamic states. In general, this task is impossible to accomplish in an exact way. It is for this reason that an approximate implementation method is required. The DDSHM is a procedure which evaluates H at a discrete number of values of the ISVs, and all other possible cases are obtained by interpolating between the computed values.

With reference to Fig. 1, the first step in the DDSHM is to solve a series of linear elastic boundary value problems (BVP) to determine the material effective elastic moduli corresponding to a *finite* number of ISV's values. In this case, one can show that the first term of Eq. (2) yields

$$\Sigma_{pq}(t) = A_{pqrs}(\lambda_i(t), \ldots, \lambda_N(t))E_{rs}(t), \quad i = 1, \ldots, N. \tag{7}$$

The fourth order tensor A_{pqrs}, which relates average strains E_{rs} to the average stresses Σ_{pq}, can be determined using a standard direct homogenization technique (Hashin, 1972) for *fixed* values of λ_i. However, it is important to note that even if the composite has elastic constituents, the effective moduli A_{pqrs} may be time dependent due to the possible presence of growing cracks. Therefore, the time dependence in Σ_{pq} arises from two sources: the (time dependent) strain history and the implicit dependence of the effective moduli on time evolving cracks.

For a fixed crack configuration, the solution to the BVP needed to define A_{pqrs} exists and is unique (Suquet, 1987). The case of an RVE with growing cracks changes the nature of the problem significantly. The problem becomes an initial-boundary value problem (IBVP) whose nature strongly depends on the evolution equations that govern the crack growth. For elastic constituents, these IBVPs have been shown to have a formal structure essentially identical to that of the quasi-static evolution of plastic flow in an elastic, perfectly plastic medium (Nguyen, 1984, 1985, 1986, 1987; Nguyen et al., 1990). The consequence is that IBVPs where a system of Griffith cracks grows quasi-statically in an elastic medium do not, in general, yield unique solutions. To define a quasi-static IBVP characterized by the existence and uniqueness of stable solutions, a more regular crack growth law must be defined by including *time dependence* (i.e., a *viscous* type behavior) in the crack growth law (Maugin, 1992). This can be accomplished by using a dissipation potential such as that in Eq. (3), which yields a crack evolution law of the following type (cf. Coussy, 1986; Schapery, 1975a,b,c):

$$\dot{\lambda}_i = \eta_i \langle G_i - G_i^{\mathrm{cr}} \rangle, \tag{8}$$

where $i = 1, \ldots, N$, and the basic material properties η_i and G_i^{cr} will be referred to as the crack growth viscosity coefficient and the critical energy release rate for the ith crack, respectively. It should be noticed that, for $\eta_i \to \infty$, Eq. (8) recovers the Griffith criterion (Griffith, 1921).

The key to the computational efficiency of the DDSHM is the manner in which the available energy release rate G_i in Eq. (8) is practically computed. With reference to Eq. (1) and the second term of Eq. (2), for a composite with elastic constituents, and in view of the so-called average virtual work theorems (cf.

Hashin, 1972; Suquet, 1985, 1987), H can be re-written in terms of the effective moduli and macroscopic strain as

$$H(t) = \frac{1}{2}A_{ijkl}(\lambda_1(t), \lambda_2(t), \ldots, \lambda_N(t))E_{ij}(t)E_{kl}(t) \tag{9}$$

for a given fixed crack configuration, *viz.*, the values of λ_i at time t. The energy release rate definition in the second term of Eq. (2) implies that the strain energy stored in the RVE must be known as a C^1 function of crack length. As mentioned earlier, this is accomplished by determining the moduli A_{ijkl} only for a finite number of fixed crack length configurations. The smooth representation needed for H is then obtained via a standard interpolation strategy (e.g. Lagrange polynomials of Press et al. (1994)).

The DDSHM constitutive and evolution equations (CEE) given by Eqs. (7) and (8) form a system of ordinary differential equations (ODEs) that describe the RVE microstructure evolution in terms of the macroscopic applied average strain history. Thus, the value of the ISVs at any point in a loading program can be determined through numerical solution of the set of ODEs indicated by Eq. (8) and not by repeatedly solving a micromechanics BVP for each load state. The DDSHM is general and subject only to those limitations associated with using an RVE approach for the determination of effective material properties. [4] Once the effective CEE have been defined, these may be used as the response functions for a homogenized material point to solve structural mechanics problems of interest.

To construct the homogenized evolution equations by the DDSHM, the effective elastic moduli must be evaluated at various stages of crack development. This implies that all possible crack paths within the chosen RVE must be known a priori. [5] This requirement is perhaps the main shortcoming of the DDSHM. However, it should be noted that even if the RVE microstructure evolution is tracked using a GL approach, some foreknowledge of the possible crack paths is still required to carry out the necessary calculations. For this reason, the authors feel that the DDSHM represents a mathematically sound and practical alternative to methods that require an explicit solution to a local micromechanics problem by FEM (or other numerical methods) for each material point as the global load history evolves. In fact, even in the presence of growing damage, the determination of the effective constitutive equations using the DDSHM is performed only once and independently of the thermo-mechanical load path that will *later* be imposed during structural analysis calculations.

In order to compare the DDSHM with traditional micromechanics based GL approach to damage modeling, the computational steps involved in modeling the response of a general structure with a constitutive law that includes crack evolution are shown in Fig. 2. Three basic steps are required regardless of the constitutive modeling approach: (i) a pre-processing phase, where a model for the local material behavior is created; (ii) a solution phase, where the global structural behavior is determined using information on the evolving material state from the constitutive law; and (iii) a post-processing phase, where material state data is stored for review.

Items $\boxed{1}$ through $\boxed{3}$ in Fig. 2 highlight the differences between the DDSHM and the GL approach. Item $\boxed{1}$ is a pre-processing step which is required by the DDSHM (cf. Eq. (7)) but not by the GL method. Note that the fact that this *extra* step is performed in the pre-processing phase, i.e., decoupled from the global solution of later structural analyses, is perhaps the most unique and desirable aspect of the DDSHM. The parameter nDPT depends on the damage space discretization, i.e., combines the number of possible crack paths as well as the number of points along each path at which the effective elastic moduli are explicitly calculated. As described in earlier works (Caiazzo and Costanzo, 2000a), this step is currently accomplished using a finite element program written especially for performing the DDSHM procedure.

[4] Among other considerations, the dimensions d_X, d_Y, and d_Z shown in Fig. 1 must be *small* relative to gradients in the macroscopic variables E_{ij} and Σ_{kl}.

[5] This requirement justifies terming this approach a DDSHM since a finite, yet possibly large, set of damage states is considered.

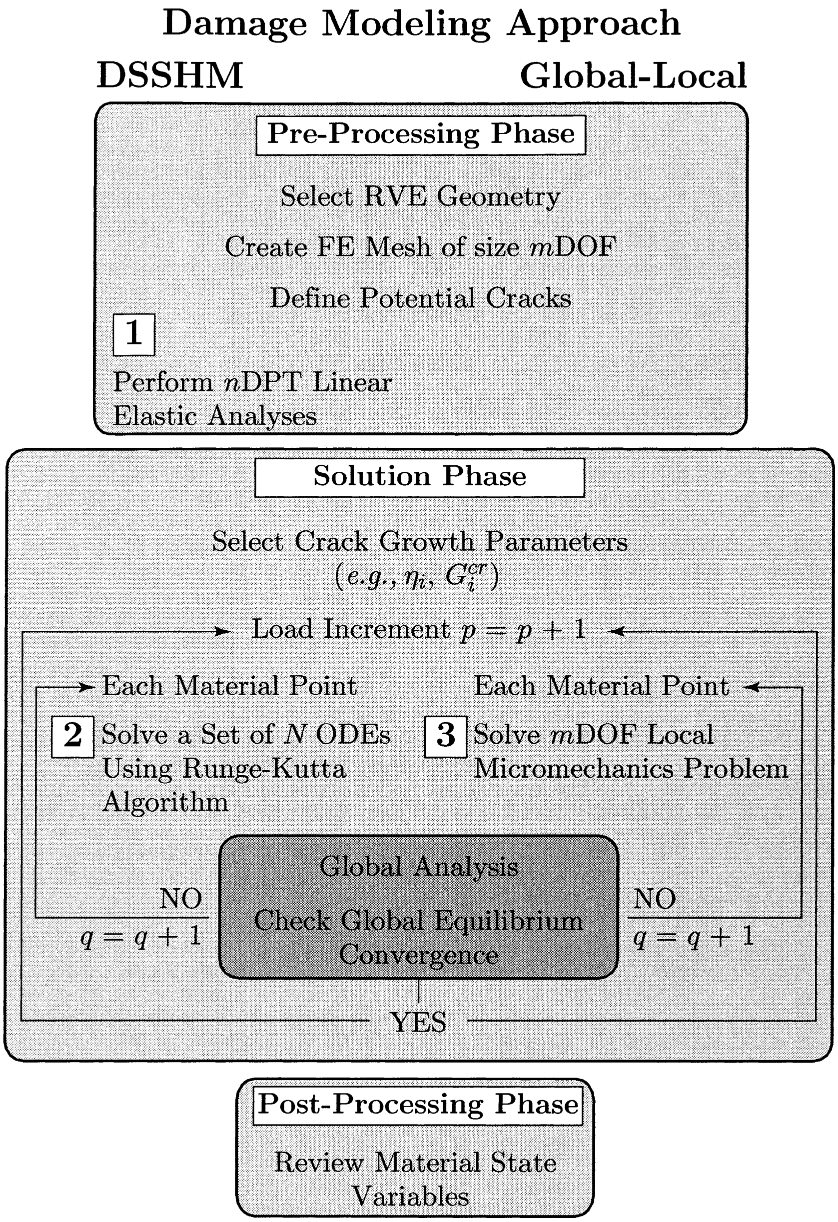

Fig. 2. A flow chart showing computational steps for a nonlinear structural analysis. Differences between the DDSHM (left) and a GL approach (right) are highlighted. The steps shown in the center are common to both the procedures.

The major computational difference between the two methods occurs during the solution phase (items 2 and 3. Crack evolution in DDSHM is determined by numerically solving a set of N ODEs indicated by Eq. (8). This is currently accomplished using a fourth order Runge–Kutta algorithm with adaptive step size control (Press et al., 1994). In the GL method, crack evolution is determined by performing a complete micromechanics analysis for each material point in the global analysis. Hence, if an identical RVE is selected for both the DDSHM and a GL analyses, we offer that the numerical solution of N ODEs will require less computational effort than solving an *m*DOF set of linear equations, where *m*DOF represents the number of degrees of freedom in the micromechanical solution of the RVE problem. In other words, for practical problems such as the computational time associated with solving $p \times q \times m$DOF, it is to be expected that *m*DOF is so large that it will considerably outweigh the DDSHM requirement of having to solve *n*DPT problems to generate the set of N ODEs which is then solved $p \times q$ times in the global analysis. It is also important to note that total solution *time* depends not only on the number of computations (related to N for DDSHM and *m*DOF for a GL approach) performed at the local level but also depends on the *stability* of the local constitutive routine. Stability here refers to the ability of the local model to return the *expected* updated stress vector and stiffness tensor for the material point to the global model such that global equilibrium is obtained in the fewest iterations possible. An example of an unstable local model is one where the stiffness and stress residual varies in a discontinuous manner for small values of state variable evolution. Because the effective moduli in the DDSHM are based upon a Lagrange interpolation of values at discrete points of crack evolution, smoothness is virtually guaranteed.

3. The effect of intraply cracks on macroscopic stiffness

Without question, the most widely researched damage propagation problem in composites is that of intraply cracking. Intraply cracks, sometimes referred to as ply or matrix microcracks, are known to develop in laminated composites subject to tensile and shear loads. Several researchers have studied the development of these cracks and measured their effect on the macroscopic stiffness. It has been offered that tracking the evolution, i.e., growth of individual intraply cracks is of no significance to understanding the composite behavior (Hashin, 1996). Rather, in all previous works documented in the literature, development or evolution is characterized by the appearance of a *new* crack surface. This can be argued intuitively based on the fact that for most practical laminates and loading conditions, the stress transverse to the fiber direction does not vary significantly with the through-the-thickness coordinate. In fact, in most of the experimental studies, cross-ply laminates are loaded in uniform tension. Thus, the measure of damage evolution is the *crack density* or number of cracks per length unit. Many researchers have used several different techniques to study how intraply cracks affect the macroscopic stiffness of composite materials (Highsmith and Reifsnider, 1982; Nairn, 1989; Hashin, 1987; Wang, 1984; Tay and Lim, 1993; Tay et al., 1997; Allen and Harris, 1987a,b). In this section, results generated using the DDSHM are compared with experimental results found in the technical literature.

As discussed earlier, the first step in the DDSHM is to generate the constitutive relation for the material RVE as a function of the ISVs. The RVE used to study the effective constitutive behavior of cross-ply laminates with intraply cracks is shown in Fig. 3. This RVE provides CEE data for several different crack densities (a/h_1) for correlation with experimental results depending upon the value of the ISVs λ_i, i.e., which of the three cracks have fully extended. Note that this 2D RVE provides only a subset of the full three-dimensional (3D) response functions and implies that crack lengths are independent of the out-of-plane coordinate. [6] This is a valid assumption for the loading types presented in the literature and to be studied here. Furthermore, since the experimental data in the literature are limited to the measurement of crack densities instead of lengths of individual microcracks, all results presented below were generated using a crack growth law that approximates the Griffith criterion. This is accomplished by setting the values of the parameters denoted by η_i in Eq. (8) to be very large. The material properties used to generate the results presented below are listed in Table 1. Said results are compared with experimental data taken from a paper

[6] A 2D RVE was chosen for computational simplicity. The DDSHM can be used to generate the response functions for a full 3D case.

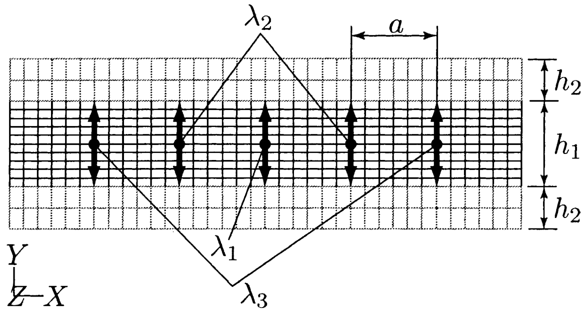

Fig. 3. A DDSHM RVE for layered composites with three crack densities.

Table 1
Material properties used in for generating the results presented in Fig. 4[a]

Property	Glass/epoxy	Carbon/epoxy
E_L (GPa)	41.7	144.8
E_T (GPa)	13.0	9.6
G_{LT} (GPa)	3.4	4.8
v_{LT}	0.30	0.31
v_{TT}	0.42	0.46
Ply thickness (mm)	0.203	0.127
Matrix fracture toughness (J/m^2)	400	800

[a] This data set is the same as that used by Lee et al. (1989).

by Lee et al. (1989) who summarized the analytical and experimental works of others for comparison with their continuum damage mechanics model. The experimental data cited by Lee et al. (1989) are repeated here to show that the DDSHM accurately models the stiffness reductions in glass and carbon fiber reinforced epoxy cross-ply laminates subjected to monotonically increasing tensile loads.

The DDSHM obtained results are summarized in Fig. 4. Note that correlation of the DDSHM with the experimental data (also presented in Fig. 4) provides no validation of the ability of the model to predict the onset of damage in the material. This correlation only shows that if the damage state is known, the DDSHM correctly determines the corresponding effective stiffness. Let us recall from Section 2 that this is only one aspect of the problem (e.g., Eq. (7)). A complete set of CEE requires that the model also predict damage evolution (e.g., Eq. (8)). Nevertheless, the excellent agreement between the DDSHM results and the experimental data does indicate that if a suitable crack growth law can be implemented, the DDSHM procedure will provide a good model of the overall constitutive and evolution response of the material. Onset and growth of damage are discussed in subsequent sections.

4. Loads to initiate damage under uniform tension

The results summarized in Fig. 4 show that the DDSHM can accurately predict the effective moduli of layered composites containing cracks as functions of damage. In this section, data are presented to show that the DDSHM is able to predict the macroscopic load state at which damage initiates given the basic information about the fracture toughness of the material.

The stress–strain response predicted by the DDSHM model of two glass and carbon fiber reinforced epoxy cross-ply laminates under monotonic tensile loading is shown in Fig. 5. These curves were obtained

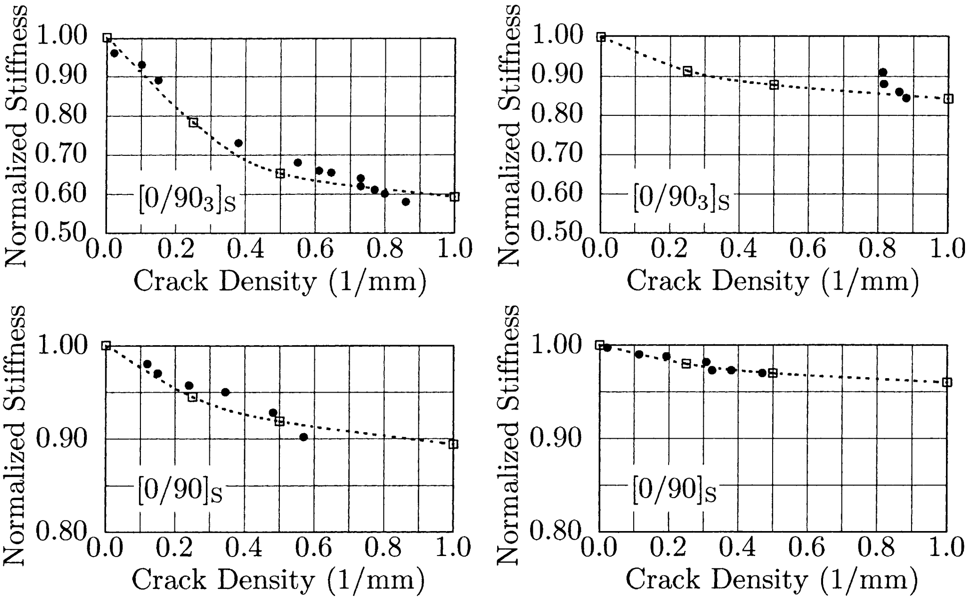

Fig. 4. Axial stiffness versus crack density for cross-ply laminates. The DDSHM predictions are shown by dashed lines. The filled circles represent the experimental data taken from the work of Lee et al. (1989). The plots in the top and bottom left quadrants concern glass/epoxy $[0/90_3]_S$ and $[0/90]_S$ cross-ply laminates, whereas plots in the top and bottom right quadrants concern results for carbon/epoxy $[0/90_3]_S$ and $[0/90]_S$ cross-ply laminates. The stiffness values have been normalized with respect to the undamaged stiffness.

using the RVE shown in Fig. 3 and the material properties in Table 1. The plots in Fig. 5 show that the DDSHM is certainly capable of qualitatively predicting the trends that have been observed experimentally by several researchers (cf. Daniel and Lee, 1990). These trends display an initial linear elastic response followed by a region where the stiffness is degrading continuously with increasing applied load as damage grows (i.e., new cracks form). Finally, there exists a region that is again linear and associated with the saturated damage state, i.e., no additional cracks form as load is increased further.

To show that the DDSHM can also provide good quantitative information, a comparison will be now presented between DDSHM predictions and experimental crack density results given by Nairn (1989) (and attributed to Highsmith and Reifsnider (1982)) concerning glass fiber reinforced epoxy cross-ply laminates. Specifically, Fig. 6 shows experimental data taken from the work of Nairn (1989) and two distinct DDSHM predictions (again using the same properties and RVE referred to so far). With reference to Eq. (8) and Fig. 3, one set of DDSHM results were generated by choosing the values for the critical energy release rates (ERR) to be the same for all cracks in the RVE. The said values were chosen to be equal to a *typical*, critical ERR value of the matrix material, i.e., 300 J/m^2 (cf. Nairn, 1989). This choice of ERRs leads the DDSHM generated constitutive and evolution equations to (a) overpredict the stress level required to develop low values of crack density and (b) to underpredict the stress level required to produce higher damage densities. This kind of result is to be expected. In fact, Hashin (1996) discusses the importance and consequences of material variability on crack development in cross-ply laminates and outlines an approach to account for these variabilities in a statistical sense. One very simple approach to simulating these effects is to assume that the critical ERR of one of the cracks is lower than that of its neighbors. This idea was used to generate the second set of the DDSHM results. Again with reference to Fig. 3, this second set of results was produced by arbitrarily setting $G_1 = 150$ J/m^2 while keeping the other critical ERR values equal to 300 J/m^2. The damage onset stress level corresponding to this choice of ERRs is lower, as expected. One could argue

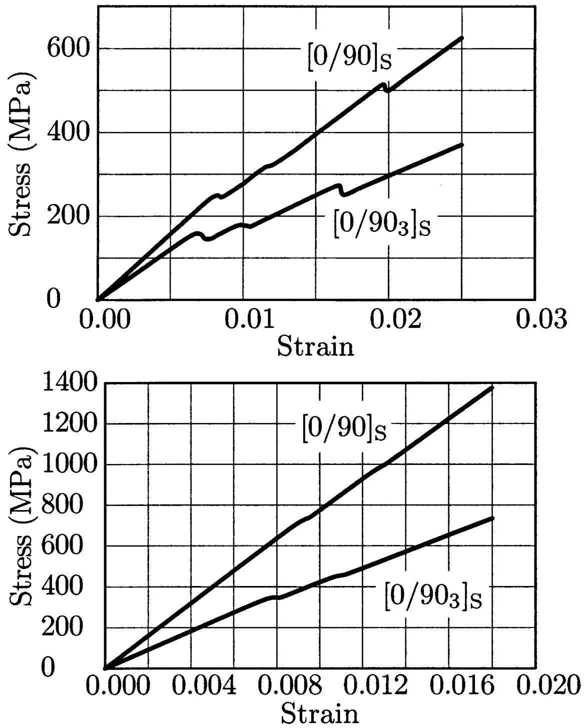

Fig. 5. Laminate tensile stress–strain curves as predicted by DDSHM. The top and bottom graphs concern results for glass/epoxy and carbon/epoxy composites, respectively.

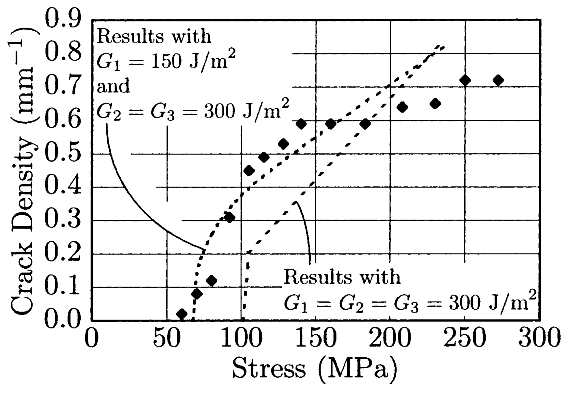

Fig. 6. Crack density as a function of the applied tensile stress for $[0/90_3]_S$ glass/epoxy laminate as predicted by the DDSHM. The filled diamonds represent the experimental results discussed by Nairn (1989).

that this *lower* ERR is a weak link in the material due to statistically varying strength properties. Once a crack develops at the *weak* location, residual and/or thermal stresses are relieved and no additional cracks form until local energy available to create new crack surfaces equals the material fracture toughness. In effect, this hypothesis of material behavior lowers the load required to develop low crack densities but does not significantly effect the load required to saturate the material with cracks.

The results shown so far indicate that the DDSHM results are generally in good agreement with the experimental data for a $[0/90_3]_S$ glass/epoxy laminate. Note that these predictions were made using basic information about the fracture toughness of the matrix material, which, for intraply crack growth, we *assumed* to be equal to the measured mode I fracture toughness. Furthermore, we have shown that to investigate the effects that G_i^{cr} has on macroscopic response (here, global stress at onset of crack growth) by the DDSHM, we need not return to the RVE analysis: we need only repeat the solution of the ODEs indicated by Eq. (8) to calculate the *new* crack evolution.

Figs. 7 and 8 show the results of similar computations and measurements for carbon fiber reinforced cross-ply laminates. Again, the DDSHM results are generally in good agreement with the measured data except, perhaps, for the case of the $[0/90]_S$ laminates wherein limited experimental data exist. Furthermore, the DDSHM predicts the physically reasonable and widely argued result that there exists a threshold region of load where no cracks form followed immediately by a rapid increase in crack density under a moderate

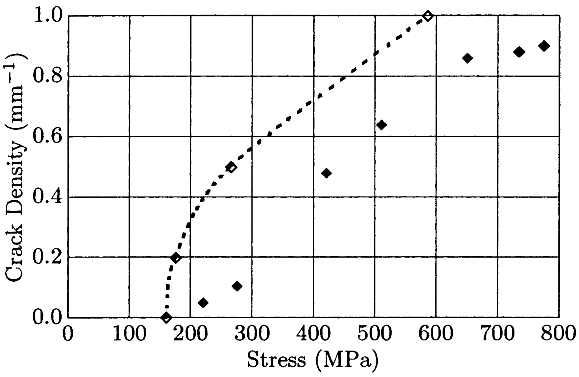

Fig. 7. Crack density as a function of applied tensile stress for $[0_2/90_2]_S$ carbon/epoxy laminate as predicted by DDSHM. The DDSHM prediction is shown by the dashed line. With reference to Fig. 3, the critical ERR for all cracks was chosen to be 80 J/m^2. The filled diamonds represent the experimental results obtained by Daniel and Lee (1990).

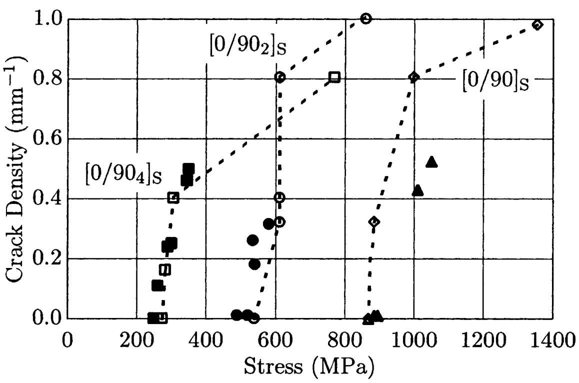

Fig. 8. Crack density as a function of applied tensile stress for several carbon/epoxy laminate as predicted by DDSHM. The DDSHM prediction is shown by the dashed lines. With reference to Fig. 3, the critical ERR for all cracks was chosen to be 800 J/m^2. The filled symbols (squares for $[0/90_4]_S$; circles for $[0/90_2]_S$; and triangles for $[0/90]_S$) represent the experimental results discussed by Hashin (1996).

increase in load. The additional load required to introduce more cracks then increases steadily for cross-ply laminates until a fiber dominated failure occurs.

5. Modeling of structural elements

A major goal of any effective continuum theory is increased computational efficiency to allow one to model the effective response of material with discrete microstructure by the smallest set of unknowns that accurately capture the material response. If the set is sufficiently small, the constitutive model can not only be used to gain an understanding of how key microstructural parameters affect the overall material response but also to model and track the onset and growth of damage in complex structures by integrating it in a structural analysis package. In this section, results of finite element analyses (FEA) of several simple structures are presented to show that the DDSHM provides a constitutive model that can easily be integrated with the ABAQUS FEA software package.

The first example presented herein considers a system that has been extensively studied by Martin (1992, 1991). Martin has developed, analyzed and tested curved beam configurations designed to investigate interlaminar and intraply damage development in unidirectional and cross-ply carbon/epoxy composite laminates. Unidirectional specimens were used to investigate interlaminar failures, while both intraply and interlaminar failures are possible in the cross-ply laminates. The rationale for presenting a comparison between the results by Martin and those obtained via FEA and DDSHM lies in the fact that previous works (cf. Sun and Kelly, 1988) indicated that predicting the onset of interply damage is difficult when using strength-based failure criteria due to the difficulty in measuring basic through-the-thickness strengths. Furthermore, this problem is also complicated by the lack of a validated failure criterion for the multi-axial stress state that occurs in the curved section of the beam specimen. Finally, the comparison is relevant to the method proposed in this paper because the analytical results obtained by Martin employed a GL scheme which relied on the application of the virtual crack closure technique for the prediction of damage evolution as opposed to an evolution law embedded in the element constitutive routine for all elements of an FEA.

Proceeding to the presentation of the results, the 2D [7] plane stress ABAQUS finite element model used to calculate the global response of the curved beam specimens is shown in Fig. 9 along with the RVE used by the DDSHM to obtain the system's effective constitutive equations. This RVE contains both intraply and interlaminar cracks. In this example, it was assumed that crack growth is symmetric within the RVE, i.e., a delamination always grows above and below the central ([0] for the unidirectional specimens and [90] for the cross-ply laminates) layer. Also, since only a single intraply crack is present within the RVE, the CEE for the cross-ply laminates correspond to a single value of crack density, viz., $a/h = 0.5$.

In Table 2, the results obtained from the DDSHM/FEA regarding the delamination onset conditions are summarized and compared with experimental and analysis data given by Martin. Additional information concerning the delamination onset is shown in Fig. 10 for the three curved beam specimens analyzed. As shown in Table 2, the location of greatest energy available to grow a delamination crack, viz. λ_3, is in excellent agreement with the results obtained by Martin (1992, 1991) using a GL analysis technique. Note that the DDSHM predictions were made by assigning a value of critical ERR (G_3^{cr}) of 80 J/m^2 to the constitutive property database for all elements in the model. This is the average value of G_{IC} for the material as reported by Martin.

[7] In preparing this paper, a simplified 2D representation of the curved beam specimen was used instead of a 3D model. Clearly, a 3D model would be needed to account for free-edge effects, and a more detailed analysis would be required to gain the level of understanding of the material behavior needed to predict structural reliability.

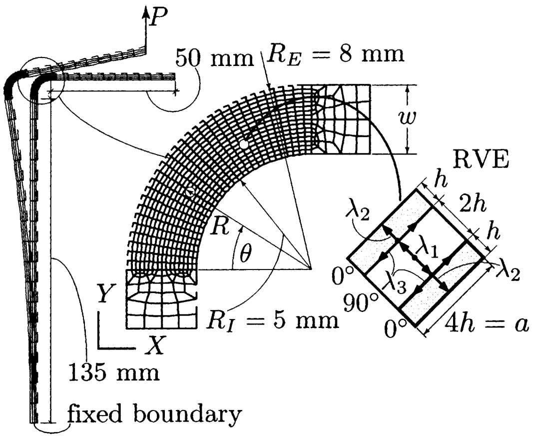

Fig. 9. The 2D ABAQUS FE model used to predict the global response of the curved beams.

Table 2
Onset of delamination results for the carbon/epoxy laminate curved beams analyzed by Martin (1991)[a]

Onset of delamination		Laminate type		
Result	Source	[0]	$[(0/90)_N]_S$	$[(0_7/90_5)_N]_S$
ρ	Martin's analysis	0.42	N/A	0.42
	Martin's experiment	0.33–0.49	N/A	N/A
	DDSHM	0.41	0.41	0.41
$\theta°$	Martin's analysis	25	N/A	25
	DDSHM	25.4	25.4	25.4
$P(N)$	Martin's experiment	7.23–14.60	N/A	6.50–10.50
	DDSHM	9.13–9.74	7.99–8.49	8.88–9.48

[a] The beam geometry is shown in Fig. 9. N/A stands for not available. With reference to Fig. 9, P and θ are defined in the figure, whereas $\rho = (R - R_I)/w$.

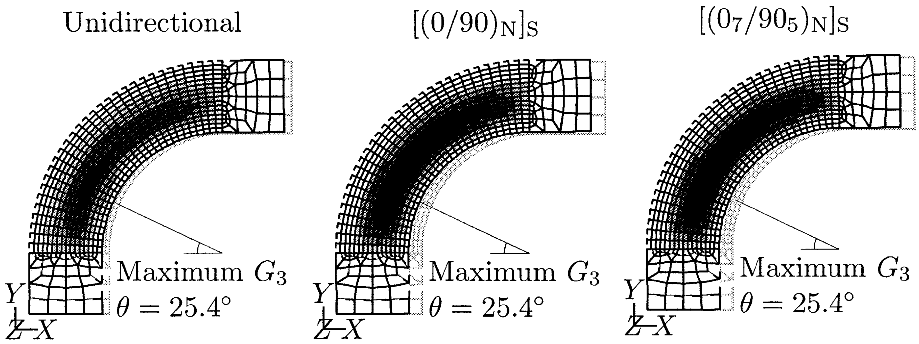

Fig. 10. RVE and graphical display of spatial variation of G_3 in the radius of the curved beam specimens.

The fact that the ρ and θ values reported in Table 2 are the same for the three lay-up configurations analyzed indicates that the point with maximum ERR is located in the same element of the underlying FE grid. In other words, the maximum ERR points for the three material configurations analyzed are close to one another. This result may appear a little surprising. However, recall that the constitutive behavior provided by the DDSHM and input into the FE analysis is that of an equivalent *homogeneous* material and not a discrete layered one. In addition, one needs to observe that the macroscopic actions (shear force and bending moment distributions) for the problem at hand are statically determined, i.e., independent of material properties. Thus, the location of the maximum through-the-thickness stress (and in this case, ERR) depends on the degree of local material anisotropy, which does not vary greatly for the materials at hand.

Another observation concerning the results given in Table 2 considers the fact that a *range* of values is reported for the critical load P instead of a single value. This is due to the fact that, for the DDSHM results, the actual value of load at which the delamination initiates is difficult to determine using a nonlinear incremental loading analysis. Rather, we are only able to establish during what load increment the interply crack *appeared* and/or extended a finite amount. A better estimate (i.e., reduced range) of the actual value of load to initiate crack growth could be obtained by reducing the maximum load increment size. However, given the significant scatter in the experimental data, no additional analyses were conducted to refine the DDSHM load predictions. Furthermore, it should be noted that the structure of the DDSHM constitutive equations allows one to investigate the effects of critical ERR on material behavior without returning to the RVE analysis. Thus, we could readily use the DDSHM CEE to determine the relative values of G_i^{cr} required to initiate each damage mode for a given global load level or history.

To substantiate the claim that the DDSHM approach is more computationally efficient and offers more freedom than a GL method, we present a final set of analysis results. With reference to the inset in Fig. 11, suppose a set of 44 ply thick ($d = 5.6$ mm) straight beams were manufactured from the $[0/90]_S$ material used for the curved beams discussed above. Furthermore, suppose that said straight beams were to be tested in three-point flexure at two different length-to-depth ratios. The CEE obtained by the DDSHM in conjunction with the curved beam analyses can be directly applied to track damage development in the three-point bending problem *without returning to the micromechanics analysis* level. To do so, we must only construct a new global FEA model of each new beam configuration. Results from this exercise are summarized in Figs. 11 and 12. Fig. 11 shows that results from the FE model using the CEE derived by the DDSHM are in good agreement with the classical Euler–Bernoulli beam theory analytical prediction for mid-span beam deflection. As expected, the overall load deflection results from the FE analysis of the *short* beam deviates slightly from the analytical result that ignores transverse shearing deformation effects. Values of the ISVs (i.e., intraply and interply crack lengths) in each element in the FE model at two load levels are plotted in Figs. 12 and 13 to show how damage develops differently in each beam configuration. A value of 1.0, shown as the darkest shading in the figures, indicates that the crack is fully extended, while lighter shades are associated with crack lengths less than fully extended. Unshaded regions are undamaged at the

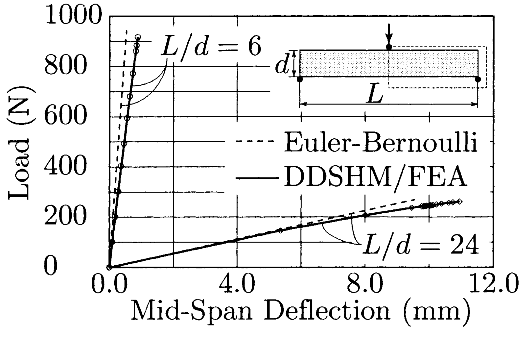

Fig. 11. Load–deflection response of three-point flexure specimen.

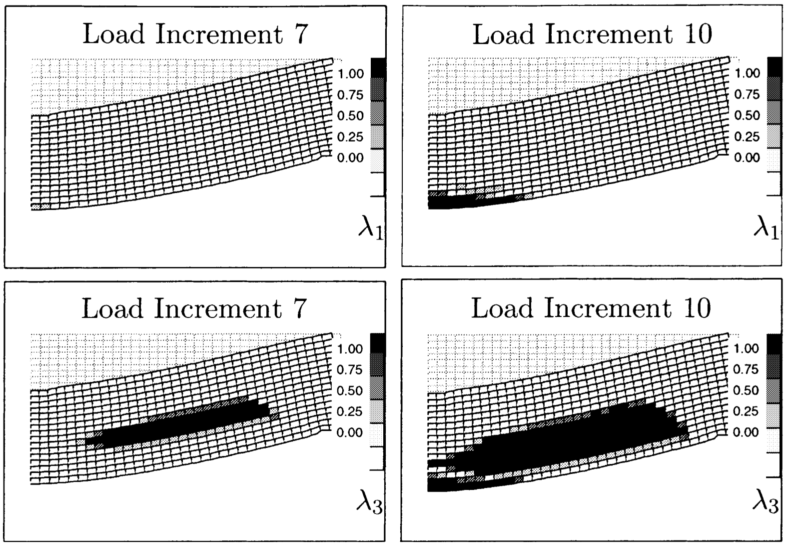

Fig. 12. Graphical display of values of ISVs in three-point flexure specimen, $L/d = 6$. Top-intraply cracks, bottom-interply cracks. The ISVs are defined in the RVE shown in Fig. 9.

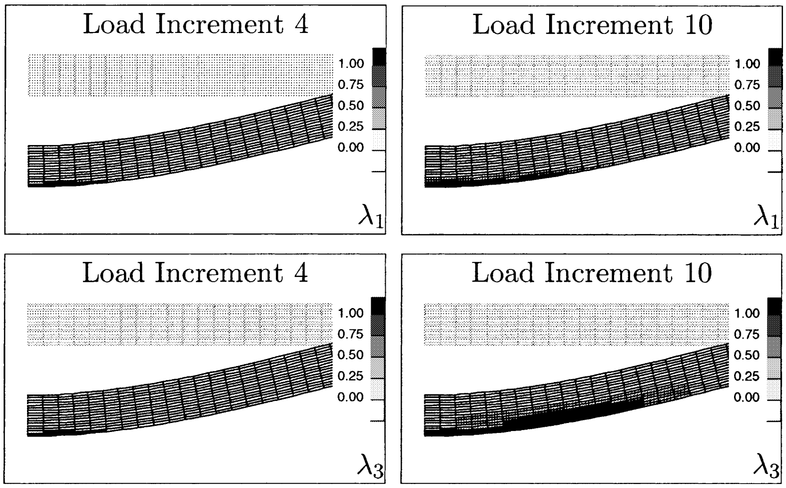

Fig. 13. Graphical display of values of ISVs in three-point flexure specimen, $L/d = 24$. Top-intraply cracks, bottom-interply cracks. The ISVs are defined in the RVE shown in Fig. 9.

load levels indicated. These analyses yield the following expected results: (i) intraply damage occurs on the tensile side of the beams, (ii) intraply damage is more widespread in the longer beam geometry, and (iii) interply cracking in the shorter specimen occurs near the middle surface of the beam where the interlaminar shear stress is the greatest. Note that the FE result that includes damage development is only slightly nonlinear since intraply cracking (and local delamination) do not significantly effect the macroscopic stiffness of the material.

6. Closing remarks

Results presented here show that the DDSHM can deliver the effective damage dependent constitutive and evolution equations of layered composite materials. The CEE derived by the DDSHM allow one to obtain good agreement with experimental data and other analysis results available in the literature, and said CEE represent the effective properties of the composite material with growing cracks in the strict sense of the expression. The DDSHM is an alternative approach to GL analysis. By definition, the GL analysis method requires that calculations be performed *locally* for each material point and load step in the analysis. In other words, GL analyses do not deliver the effective constitutive relations for a material. Rather, they provide the value of the current average stress (or strain) for the current value of average strain (or stress) at the material point. In fact, GL analysis methods do not provide any information that can be used in any other analysis than the particular problem at hand, e.g., there is no way to apply the virtual crack closure results from a curved beam analysis to the three-point flexure test on the same material. Clearly, a more desirable result would be one where the *local* (RVE) effective constitutive and damage evolution equations for a given material could be obtained *once and for all* and *separately* from the *global* analysis. The DDSHM precisely delivers this information. Furthermore, as shown here, the DDSHM CEE are derived in a form that is easily integrated with a global structural analysis package and therefore can be applied to a wide range of structures.

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